(1) $\dfrac{x}{x+1} < \dfrac{2}{x+2} \Rightarrow \dfrac{x(x+2) - 2(x+1)}{(x+1)(x+2)} < 0 \Rightarrow \dfrac{x^2 - 2}{(x+1)(x+2)} < 0$ [6]

c.v's are $-2, \pm\sqrt{2}, -1$.

$x < -2$	$-2 < x < -\sqrt{2}$	$-\sqrt{2} < x < -1$	$-1 < x < \sqrt{2}$	$x > \sqrt{2}$
>0	<0	>0	<0	>0

so $\{x \in \mathbb{R} \mid -2 < x < -\sqrt{2}\} \cup \{x \in \mathbb{R} \mid -1 < x < \sqrt{2}\}$

② ⓐ $\dfrac{(r-3)(r+1)(r+2) + (r+2) - (r+1)}{(r+1)(r+2)} = \dfrac{(r^2+3r+2)(r-3) + 1}{(r+1)(r+2)}$ [2]

$= \dfrac{r^3 - 3r^2 + 3r^2 + 2r - 9r - 6 + 1}{(r+1)(r+2)} = \dfrac{r^3 - 7r - 5}{(r+1)(r+2)} \equiv r - 3 + \dfrac{1}{r+1} - \dfrac{1}{r+2}$

ⓑ $\displaystyle\sum_{r=1}^{n} \frac{r^3 - 7r - 5}{(r+1)(r+2)} \equiv \sum_{r=1}^{n} r - 3 + \frac{1}{r+1} - \frac{1}{r+2} = \frac{1}{2}n(n+1) - 3n + \sum_{r=1}^{n} \frac{1}{r+1} - \frac{1}{r+2}$

where the latter sum is $\dfrac{1}{2} - \dfrac{1}{3}$ $= \dfrac{1}{2} - \dfrac{1}{n+2} = \dfrac{n}{2(n+2)}$. [5]

$\dfrac{1}{3} - \dfrac{1}{4}$

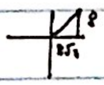
$+ \cdots$

$+ \dfrac{1}{n} - \dfrac{1}{n+1}$

$+ \dfrac{1}{n+1} - \dfrac{1}{n+2}$

So $\dfrac{1}{2}n(n+1) - \dfrac{6n}{2} + \dfrac{n}{2(n+2)} = \dfrac{n(n+1)(n+2) - 6n(n+2) + n}{2(n+2)} = \dfrac{n}{2(n+2)}(n^2 + 3n + 2 - 6n - 12 + 1)$

$= \dfrac{n}{2(n+2)}(n^2 - 3n - 9)$ //

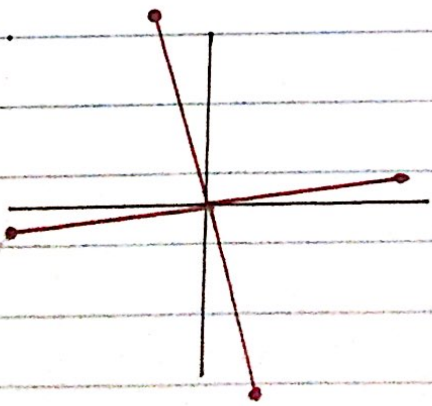
③ ⓐ 

so $\arg(8\sqrt{3} + 8i) = \dfrac{\pi}{6}$. $|8\sqrt{3} + 8i| = 16$. [5]

So $z^4 = 8\sqrt{3} + 8i \Rightarrow z^4 = 16e^{i(\frac{\pi}{6} + 2k\pi)} \Rightarrow z = 2e^{i(\frac{\pi}{24} + \frac{\pi k}{2})}$

$\therefore z = 2e^{i\pi/24}, 2e^{i13\pi/24}, 2e^{25i\pi/24}, 2e^{37i\pi/24}$.

ⓑ [2]



④ (i) $p \frac{dx}{dt} + qx = r. \overset{(p \neq 0)}{\Longleftrightarrow} \frac{dx}{dt} + \frac{q}{p}x = \frac{r}{p}.$

ⓐ Find x in terms of t (given $x=0, t=0$). [4]

Integrading factor ~~e^{...}~~ $e^{qt/p}$

So $x e^{qt/p} = \frac{r}{p} \cdot \frac{p}{q} e^{qt/p} + c.$

$(0,0)$: $0 = \frac{r}{q} + c \Rightarrow c = -r/q.$

$\therefore x = \frac{r}{q}\left(1 - e^{-qt/p}\right).$ //

ⓑ as $t \to \infty$ we have $e^{-qt/p} \to 0$ so $x \to \frac{r}{q}.$ // [1]

(ii) $\frac{dy}{d\theta} + 2y = \sin\theta.$ Complimentary function : $\frac{dy}{d\theta} = -2y$ [7]

So $\ln y = -2\theta + c \Rightarrow y_{CF} = Ae^{-2\theta}.$

Guess a particular integral $y_{PI} = a\sin\theta + b\cos\theta$

So $a\cos\theta - b\sin\theta + 2a\sin\theta + 2b\cos\theta = \sin\theta$

$\Rightarrow \sin\theta\,(2a - b) + \cos\theta\,(2b + a) = \sin\theta$

equading coefficients : $2a - b = 1$ ① so $2 \times$ ① $+$ ② : $5a = 2$

$2b + a = 0$ ② $\Rightarrow a = 2/5.$

$\Rightarrow b = -1/5.$

So $y = y_{CF} + y_{PI} = Ae^{-2\theta} + \frac{2}{5}\sin\theta - \frac{1}{5}\cos\theta$

Given $(0,0)$: $0 = A - \frac{1}{5} \Rightarrow A = 1/5.$

$\therefore y = \frac{1}{5}\left(e^{-2\theta} + 2\sin\theta - \cos\theta\right)$ //

⑤ (a) $\sin^5\theta = a\sin 5\theta + b\sin 3\theta + c\sin\theta.$ [5]

Recall that ~~sin θ~~ if $z = \cos\theta + i\sin\theta$ then $z - z^{-1} = 2i\sin\theta.$

So $(2i\sin\theta)^5 = (z - z^{-1})^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$

$= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right) = 2i\sin 5\theta$ ~~− 10i cos~~

$= 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$

So $32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$

$\Rightarrow \sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$ //

(b) $\int_0^{\pi/3} \sin^5\theta \, d\theta = \frac{1}{16}\int_0^{\pi/3} \sin 5\theta - 5\sin 3\theta + 10\sin\theta \, d\theta$ [5]

$= \frac{1}{16}\left[-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right]_0^{\pi/3}$

$= \frac{1}{16}\left[-\frac{1}{10} - \frac{5}{3} - 5 + \frac{1}{5} - \frac{5}{3} + 10\right]$

$= \frac{1}{16}\left[\frac{53}{30}\right] = \frac{53}{480}$ //

⑥ (a) Let $f(x) = \tan x$. $\quad f(\pi/4) = 1$. [7]

$f'(x) = \sec^2 x \Rightarrow f'(\pi/4) = 2$.

$f''(x) = 2\sec x \cdot \sec x \tan x \Rightarrow f''(\pi/4) = 4$

$f'''(x) = \frac{d}{dx}(2\sec^2 x \tan x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$\Rightarrow f'''(\pi/4) = 16$.

so $f(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$ //

(b) Set $x = \frac{5\pi}{12}$: $\tan\frac{5\pi}{12} \approx 1 + 2\left(\frac{5\pi}{12} - \frac{\pi}{4}\right) + 2\left(\frac{5\pi}{12} - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(\frac{5\pi}{12} - \frac{\pi}{4}\right)^3$

$= 1 + \frac{\pi}{3} + \frac{\pi^2}{18} + \frac{\pi^3}{81}$. [2]

⑦ (a) Sub $x = e^u$ into $x^2 y'' - 2xy' + 2y = -x^{-2}$. [6]

So $\frac{dx}{du} = e^u \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot e^{-u}$.

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du} \cdot e^{-u}\right) = \frac{d}{du}\left(\frac{dy}{du} \cdot e^{-u}\right) \cdot \frac{du}{dx} = e^{-u}\left(\frac{d^2y}{du^2} e^{-u} - e^{-u}\frac{dy}{du}\right)$

$= e^{-2u}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)$.

So $e^{2u} \cdot e^{-2u}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) - 2e^u e^{-u}\frac{dy}{du} + 2y = -e^{-2u}$

$\Leftrightarrow \frac{d^2y}{du^2} - 3\frac{dy}{du} + 2y = -e^{-2u}$ // as required.

(b) Aux equation: $\lambda^2 - 3\lambda + 2 = 0 \Leftrightarrow (\lambda - 2)(\lambda - 1) = 0 \Leftrightarrow \lambda = 2$ or $\lambda = 1$

So $y_{CF} = Ae^{2u} + Be^u$. [7]

guess $y_{PI} = me^{-2u} \Rightarrow y' = -2me^{-2u} \Rightarrow y'' = 4me^{-2u}$.

So $4me^{-2u} + 6me^{-2u} + 2me^{-2u} = -e^{-2u} \Rightarrow 12m = -1 \Rightarrow m = \frac{-1}{12}$.

$\therefore y = Ae^{2u} + Be^u - \frac{1}{12}e^{-2u}$ //

ⓒ $y = Ax^2 + Bx - \frac{1}{12x^2}$. [1]

⑧ ⓐ Intersect at $7\cos\theta = 3 + 3\cos\theta \Leftrightarrow \cos\theta = 3/4$. [3]

so $P\left(\frac{21}{4}, \arccos\frac{3}{4}\right)$, $Q\left(\frac{21}{4}, -\arccos\frac{3}{4}\right)$.

ⓑ By symmetry we only need to find the top half and then double. Strategy is to integrate C_2 from 0 to arccos 3/4 and then C_1 from arccos 3/4 to $\pi/2$. [7]

For convenience: $\alpha = \arccos 3/4$: $9\int_0^{\alpha} 1 + 2\cos\theta + \cos^2\theta \, d\theta$

$= 9\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\alpha} = 9\left[\frac{3\alpha}{2} + 2\times\frac{1}{4}\times\sqrt{7} + \frac{1}{4}\times\frac{3}{8}\times\sqrt{7}\right]$

$= 9\left[\frac{3\alpha}{2} + \frac{19}{32}\sqrt{7}\right] = 9\left[\frac{3\alpha}{2} + \frac{19}{32}\sqrt{7}\right]$

And $\int_{\alpha}^{\pi/2} 49/2\,(1 + \cos 2\theta)\, d\theta = \frac{49}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\alpha}^{\pi/2}$

$= \frac{49}{2}\left[\frac{\pi}{2} - \alpha - \frac{3}{16}\sqrt{7}\right]$.

So adding: $R = \frac{49\pi}{4} - 11\alpha$ ~~$+ \frac{165}{32}\sqrt{7}$~~ $+ \frac{3}{4}\sqrt{7}$

≈ 32.5 (3 s.f)

(exact: $\frac{49\pi}{4} - 11\arccos 3/4 + \frac{3\sqrt{7}}{4}$).